Linear Algebra and its Applications 436 (2012) 1114–1121

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Balance in gain graphs – A spectral analysis

Shahul Hameed K [a,*], K.A. Germina [b]

[a] Department of Mathematics, Government Brennen College, Thalassery 670 106, India
[b] Research Center & PG Department of Mathematics, Mary Matha Arts & Science College, Vemom P.O., Mananthavady 670 645, India

ARTICLE INFO

Article history:
Received 23 June 2011
Accepted 5 July 2011
Available online 13 October 2011

Submitted by R.A. Brualdi

AMS classification:
05C22
05C50
05C76

Keywords:
Gain graph
Signed graph
Graph characteristic polynomials

ABSTRACT

A gain graph is a graph where the edges are given some orientation and labeled with the elements (called gains) from a group so that gains are inverted when we reverse the direction of the edges. A signed graph with labels from the multiplicative group $\{1, -1\}$ on the edges can be taken as a particular case of a gain graph. In this article, we initiate a matrix analysis of gain graphs by defining the adjacency matrices of gain graphs when the underlying group for labeling the edges is the multiplicative group of a field and characterize balance in such gain graphs using the characteristic polynomials. We also establish recurrence relations for the characteristic polynomials of a gain graph and discuss consequences of these relations.

1. Introduction

Spectra of graphs (see [2]) and those of signed graphs (see [1,3,4,11–13]) have been discussed at length in the literature. A gain graph, as generalization of a signed graph, can be defined formally as $\Phi = (G, \Gamma, \phi)$ where $G = (V, \overrightarrow{E})$ is a graph with some prescribed orientation for the edges and $\phi : \overrightarrow{E} \rightarrow \Gamma$ is an edge labeling function to the group Γ with the assumption that for $\overrightarrow{v_iv_j} \in \overrightarrow{E}$, $\phi(\overrightarrow{v_jv_i}) = (\phi(\overrightarrow{v_iv_j}))^{-1}$ (For more details see [13]). In this paper, we introduce the adjacency matrix of a gain graph when the underlying group is the multiplicative group of a field and analyze balance in such gain graphs with the help of characteristic polynomial of the adjacency matrix and give recurrence relations for the computation of characteristic polynomials of such gain graphs. We denote the underlying group by $F^\times$, where F is a field and $F^\times = F \setminus \{0_F\}$ is the multiplicative group of F. The adjacency matrix $A(\Phi) = (a_{ij})$, of a gain graph $\Phi = (G, F^\times, \phi)$, is defined as

* Corresponding author.
E-mail addresses: shabrennen@gmail.com (Shahul Hameed K), srgerminaka@gmail.com (K.A. Germina).

0024-3795/$ - see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2011.07.005

$$a_{ij} = \begin{cases} \phi(v_iv_j), & \text{if } \overrightarrow{v_iv_j} \in \overrightarrow{E} \\ 0_F, & \text{otherwise.} \end{cases}$$

and $a_{ji} = (\phi(v_iv_j))^{-1}$ where 0_F is the additive identity in the field. If there is no scope for confusion we drop the subscript F and write the additive identity in F simply as 0 and the multiplicative identity of F as 1. The *characteristic polynomial* of Φ, denoted by $\Psi(\Phi, x)$ or simply as $\Psi(\Phi)$, is defined as the characteristic polynomial $\det(xI - A(\Phi)) \in F[x]$ of $A(\Phi)$, where I is the identity matrix of order as that of $A(\Phi)$. The zeros of $\Psi(\Phi, x)$, in F or in its splitting field, are called *eigenvalues* of Φ. For more details on matrices with entries from a field, their determinants, eigenvalues and eigenvectors, the reader may refer to [8]. All the underlying graphs in this paper are simple and loop-free and basic definitions regarding graphs are taken as in [6].

A signed graph can be taken as a gain graph with the underlying group as $GF(3)^\times = \{1, 2\}$ or as the multiplicative subgroup $\{1, -1\}$ of $\mathbb{R}^\times$ and a graph as the gain graph with the underlying group as $GF(2)^\times = \{1\}$. The *gain* $\phi(C)$, of a cycle $C : v_0v_1 \ldots v_nv_0$, is the product $\phi(v_0v_1)\phi(v_1v_2)\ldots\phi(v_nv_0)$ of gains of its edges. A gain graph $\Phi = (G, \Gamma, \phi)$ is said to be *cycle balanced* or simply *balanced* if $\phi(C) = 1$ for all cycles C in it. Also, when the underlying graph is a path P_n, we call the corresponding gain graph to be *gain path* and when the underlying graph is a cycle C_n, the gain graph is called *gain cycle*. We define a *linear subgraph L* of Φ as a subgraph of the underlying graph G with gains from Φ itself whose components are either K_2 or cycles. Throughout this paper, unless otherwise mentioned, $K(L)$ denotes the number of components in L and $C(L)$ denotes the number of cyclic components in L.

2. Spectral characterization of balance in gain graphs

In the case of a signed graph $\Sigma = (G, \{1, -1\}, \sigma)$, the following theorem due to Acharya is well-known.

Theorem 2.1 [1]. *Σ is a balanced signed graph if, and only if, both Σ and its underlying graph G have the same characteristic polynomials.*

Before we extend the same to the case of gain graphs over $F^\times$, we require some results, which we do in the following two theorems. We denote by $\mathfrak{L}_m$ the set of linear subgraphs of order m. To avoid further ambiguity about $\mathfrak{L}_0$ and $\mathfrak{L}_1$, in the following theorem, we take $a_0(\Phi) = 1$ and $a_1(\Phi) = \operatorname{trace}(A(\Phi)) = 0$. Also in the proof of the following result, the method of associating determinant of a matrix with that of a weighted graph is standard and much of the details are omitted which the reader may find in [9].

Theorem 2.2. *If $\Phi = (G, F^\times, \phi)$ is a gain graph where $G = (V, E)$ is a graph of order n, and if $\Psi(\Phi, x) = \sum_{i=0}^{n} a_i(\Phi)x^{n-i}$ then*

$$a_i(\Phi) = \sum_{L \in \mathfrak{L}_i} (-1)^{K(L)} \prod_{C \in L} (\phi(C) + \phi^{-1}(C)) \tag{1}$$

Proof. If $M = (a_{ij})$ is a square matrix of order m with entries from a field F, then

$$\det(M) = \sum_{\pi \in S_m} \operatorname{sgn}\pi \; a_{1\pi(1)}a_{2\pi(2)} \ldots a_{m\pi(m)} \tag{2}$$

where π is a permutation in the symmetric group S_m on the set of elements $\{1, 2, \ldots, m\}$. In the case when $M = xI - A(\Phi)$, a square matrix of order n, since the underlying graph G is simple and the fact that the coefficients $a_i(\Phi)$ of x^{n-i} are $(-1)^i$ times the principal minors of order $n-(n-i)=i$ in the expansion of $\det(xI-A(\Phi))$, from Eq. (2) and the above remarks,

$$a_i(\Phi)=(-1)^i(-1)^i\sum_{L\in\mathfrak{L}_i}(-1)^{K(L)}\prod_{\overrightarrow{e}\in E(L)}\phi(\overrightarrow{e})=\sum_{L\in\mathfrak{L}_i}(-1)^{K(L)}\prod_{\overrightarrow{e}\in E(L)}\phi(\overrightarrow{e})$$

Since every linear subgraph L consists of only K_2 or cycles,

$$\prod_{\overrightarrow{e}\in E(L)}\phi(\overrightarrow{e})=\prod_{K_2\in L}1\prod_{C\in L}(\phi(C)+\phi^{-1}(C))=\prod_{C\in L}(\phi(C)+\phi^{-1}(C))$$

and hence the formula in Eq. (1). □

Corollary 2.3. *If $\Phi=(G,F^\times,\phi)$ is a gain graph where $G=(V,E)$ is a graph of order n, then*

$$\det(A(\Phi))=(-1)^n\sum_{L\in\mathfrak{L}_n}(-1)^{K(L)}\prod_{C\in L}(\phi(C)+\phi^{-1}(C)) \tag{3}$$

Proof. Proof is immediate since $\det(A(\Phi))=(-1)^n\Psi(\Phi,0)=(-1)^n a_n(\Phi)$. □

Now we are ready to characterize balance in gain graphs.

Theorem 2.4. *If $\Phi=(G,F^\times,\phi)$ is a gain graph where $G=(V,E)$, then Φ is balanced if, and only if, $\Psi(\Phi,x)=\Psi(G,x)$.*

Proof. First we note the fact that the underlying graph G can be considered as the gain graph with all edge gains as 1. This gives, with the help of Eq. (1),

$$a_i(G)=\sum_{L\in\mathfrak{L}_i}(-1)^{K(L)}2^{C(L)}. \tag{4}$$

This formula matches exactly with the formula for ordinary graph G given in [7]. Now suppose that Φ is balanced. Then $\phi(C)=1=(\phi(C))^{-1}$ for all cycles C of Φ. Then from Eqs. (1) and (4), we get $a_i(\Phi)=a_i(G)$ for all i and hence $\Psi(\Phi,x)=\Psi(G,x)$. Conversely, assume that $\Psi(\Phi,x)=\Psi(G,x)$. Then from Eqs. (1) and (4) again,

$$\sum_{L\in\mathfrak{L}_i}(-1)^{K(L)}\prod_{C\in L}(\phi(C)+\phi^{-1}(C))=\sum_{L\in\mathfrak{L}_i}(-1)^{K(L)}2^{C(L)}$$

which gives

$$\prod_{C\in L}(\phi(C)+\phi^{-1}(C))=2^{C(L)}$$

for all linear subgraphs L. This, in turn, leads to

$$\phi(C)+\phi^{-1}(C)=2$$

for all cycles C, making the gain graph Φ to be balanced. □

3. Recurrence relations for the characteristic polynomial of a gain graph

Gill and Acharya [5] generalized the recurrence formula, given by Schwenk [10], for the computation of characteristic polynomial of graphs to signed graphs. We further extend these recurrence formula for the computation of characteristic polynomial of gain graph over $F^\times$. From now onwards, to emphasize the underlying graph G for the gain graph $\Phi = (G, F^{\times}, \phi)$, we denote the gain graph by Φ_G. Moreover, if G' is a subgraph of G, then $\Phi_{G'}$ denotes the gain graph with the underlying graph G' and the gain function ϕ restricted to the edge set of G'.

Theorem 3.1. *If $\Phi = (G, F^{\times}, \phi)$ is a gain graph where $G = (V, E)$ is a graph of order n and if v is an arbitrary vertex of G, then*

$$\Psi(\Phi_G, x) = x\Psi(\Phi_{G-v}, x) - \sum_{w \sim v} \Psi(\Phi_{G-\{v,w\}}, x) - \sum_{C \in \mathcal{C}(v)} (\phi(C) + (\phi(C))^{-1})\Psi(\Phi_{G-V(C)}, x) \quad (5)$$

where $\mathcal{C}(v)$ denotes the set of all cycles containing v.

Proof. Let $\Psi(\Phi_G, x) = \sum_{i=0}^{n} a_i(\Phi)x^{n-i}$. Then from Eq. (1),

$$a_i(\Phi) = \sum_{L \in \mathfrak{L}_i} (-1)^{K(L)} \prod_{C \in L} (\phi(C) + \phi^{-1}(C)) \quad (6)$$

Now, for any $L \in \mathfrak{L}_i$, with respect to the arbitrarily chosen vertex v, three cases are to be considered, namely,

Case (1): $v \notin L$. Here we let L' as the same linear subgraph L and view it as a linear subgraph for Φ_{G-v}.
Case (2): v belongs to an acyclic component of L whence we take L' as $L' = L - V(\overrightarrow{K_2})$ which will be viewed as the linear subgraph of $\Phi_{G-\{v,w\}}$ such that w is adjacent to v on $\overrightarrow{K_2}$.
Case (3): v belongs to a cycle component $C = C(v)$ of L whence we take L' as $L' = L - V(C)$ which is viewed as a linear subgraph of $\Phi_{G-V(C)}$.

Thus there is a one–one correspondence between the terms on the left and those on the right of Eq. (5). Also, if L contributes an amount m towards $a_i(\Phi)$ on the left, we claim below that on the right L' also contributes m to some term in each of the following cases:

Case (1): Since $L' = L$, we see that L' contributes m to the coefficient of x^{n-1-i} in $\Psi(\Phi_{G-v})$ and thus supplies m toward the coefficient of x^{n-i} in $x\Psi(\Phi_{G-v})$.
Case (2): In this case $L' = L - V(\overrightarrow{K_2})$. So L' contributes

$$\begin{aligned}(-1)^{K(L')} \prod_{C \in L'} (\phi(C) + \phi^{-1}(C)) &= (-1)^{K(L)-1} \prod_{C \in L} (\phi(C) + \phi^{-1}(C)) \\ &= -m.\end{aligned}$$

to the coefficient of $x^{n-2-(i-2)} = x^{n-i}$ in $\Psi(\Phi_{G-\{v,w\}})$ where w is the point of $\overrightarrow{K_2}$ and hence it supplies m to the coefficient of x^{n-i} in $-\Psi(\Phi_{G-\{v,w\}})$.
Case (3): In this case consider a particular cyclic component Z in a linear subgraph L where v is a vertex on it. As we have $L' = L - V(Z)$, it contributes

$$\begin{aligned}&(-1)^{K(L')} \prod_{C \in L'} (\phi(C) + \phi^{-1}(C)) \\ &= (-1)^{K(L)-1} \prod_{C \in L} (\phi(C) + \phi^{-1}(C))(\phi(Z) + \phi^{-1}(Z))^{-1} \\ &= -m(\phi(Z) + \phi^{-1}(Z))^{-1}.\end{aligned}$$

to the coefficient of $x^{n-t-(i-t)} = x^{n-i}$ in $\Psi(\Phi_{G-V(Z)})$ where t is the length of the cycle Z and hence it supplies m to the coefficient of x^{n-i} in $-(\phi(Z)+\phi^{-1}(Z))\Psi(\Phi_{G-V(Z)})$ where Z is a cycle containing v. $\square$

Since in a signed graph Σ, considered as a gain graph over the multiplicative group of the field $\{0, 1, -1\}$, on any balanced or unbalanced cycle C, we have $\phi(C) + (\phi(C))^{-1} = 2$ or -2 respectively. Therefore the above recurrence formula gives the same result as in [5] which we particulary give in the following Corollary.

Corollary 3.2. *If $\Sigma = (G, \phi)$ is a signed graph where $G = (V, E)$ is a graph of order n and if v is an arbitrary vertex of G, then*

$$\begin{aligned}\Psi(\Sigma_G, x) = x\Psi(\Sigma_{G-v}, x) - \sum_{w\sim v} \Psi(\Sigma_{G-\{v,w\}}, x)\\ -2\left(\sum_{C\in\mathcal{B}(v)} \Psi(\Sigma_{G-V(C)}, x) - \sum_{C\in\mathcal{B}'(v)} \Psi(\Sigma_{G-V(C)}, x)\right)\end{aligned}$$

where $\mathcal{B}(v)$ denotes the set of all balanced cycles containing v and $\mathcal{B}'(v)$ denotes the set of all unbalanced cycles containing v .

To get the recurrence relation in the case of a graph G as given in [10], we note that all cycles in G are balanced and hence we have the following Corollary.

Corollary 3.3. *If $G = (V, E)$ is a graph of order n and if v is an arbitrary vertex of G, then*

$$\Psi(G, x) = x\Psi(G - v, x) - \sum_{w\sim v} \Psi(G - \{v, w\}, x) - 2 \sum_{C\in\mathcal{C}(v)} \Psi(G - V(C), x)$$

where $\mathcal{C}(v)$ denotes the set of all cycles containing v.

3.1. Recurrence relation for the characteristic polynomial of gain paths and gain cycles

As a path does not contain any cycle, every gain path is obviously cycle balanced. Hence from Theorem 2.4 and the general recurrence formula we easily get the recurrence formula for any gain path in the following form.

Corollary 3.4. *If $\Phi_{P_n} = (P_n, F^\times, \phi)$ is a gain path, where $F^\times$ is the multiplicative group of a finite or infinite field F, then*

$$\Psi(\Phi_{P_n}, x) = x\Psi(\Phi_{P_{n-1}}, x) - \Psi(\Phi_{P_{n-2}}, x) \tag{7}$$

Though it is easy to deduce results for gain cycles also from the above given general recurrence formula (5) for any gain graph, we provide here independent direct proof for the recurrence relations in case of the characteristic polynomial of gain cycles.

Theorem 3.5. *If $\Phi_{C_n} = (C_n, F^\times, \phi)$ is a gain cycle, then*

$$\Psi(\Phi_{C_n}, x) = x\Psi(\Phi_{P_{n-1}}, x) - 2\Psi(\Phi_{P_{n-2}}, x) - (\phi(C_n) + (\phi(C_n))^{-1}). \tag{8}$$

or equivalently,

$$\Psi(\Phi_{C_n}, x) = \Psi(\Phi_{P_n}, x) - \Psi(\Phi_{P_{n-2}}, x) - (\phi(C_n) + (\phi(C_n))^{-1}). \tag{9}$$

Proof. Let $C_n : v_1e_1v_2e_2v_3 \cdots v_{n-1}e_{n-1}v_ne_nv_1$ be the underlying cycle for Φ_{C_n}. Then,

$$\Psi(\Phi_{C_n}) = \begin{vmatrix} x & -\phi(e_1) & 0 & 0 & \dots & 0 & -\phi(e_n)^{-1} \\ -\phi(e_1)^{-1} & x & -\phi(e_2) & 0 & \dots & 0 & 0 \\ 0 & -\phi(e_2)^{-1} & x & -\phi(e_3) & \dots & 0 & 0 \\ 0 & 0 & -\phi(e_3)^{-1} & x & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \dots & x & -\phi(e_{n-1}) \\ -\phi(e_n) & 0 & \dots & \dots & \dots & -\phi(e_{n-1})^{-1} & x \end{vmatrix}_{n\times n}$$

$$= x\Psi(\Phi_{P_{n-1}}, x)$$

$$+\phi(e_1) \begin{vmatrix} -\phi(e_1)^{-1} & -\phi(e_2) & 0 & 0 & \dots & 0 \\ 0 & x & -\phi(e_3) & \dots & 0 & 0 \\ 0 & -\phi(e_3)^{-1} & x & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & x & -\phi(e_{n-1}) \\ -\phi(e_n) & 0 & \dots & \dots & -\phi(e_{n-1})^{-1} & x \end{vmatrix}_{n-1\times n-1}$$

$$-\phi(e_n)^{-1} \begin{vmatrix} -\phi(e_1)^{-1} & x & -\phi(e_2) & 0 & \dots & 0 \\ 0 & -\phi(e_2)^{-1} & x & -\phi(e_3) & \dots & 0 \\ 0 & 0 & -\phi(e_3)^{-1} & x & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & -\phi(e_{n-2})^{-1} & x \\ -\phi(e_n) & 0 & \dots & \dots & \dots & -\phi(e_{n-1})^{-1} \end{vmatrix}_{n-1\times n-1}$$

Expanding along the first columns,

$$\Psi(\Phi_{C_n}) = x\Psi(\Phi_{P_{n-1}}, x) - \Psi(\Phi_{P_{n-2}}, x) - \Psi(\Phi_{P_{n-2}}, x) - (\phi(C_n) + (\phi(C_n))^{-1})$$

which on simplification leads to the desired recurrence relation in Eq. (8) and the equivalent Eq. (9) is due to the recurrence relation for gain paths given in Eq. (7). $\square$

Corollary 3.6. *If $\Phi_{C_n} = (C_n, F^\times, \phi)$ is a gain cycle, then*

$$\det(A(\Phi_{C_n})) = \begin{cases} \phi(C_n) + (\phi(C_n))^{-1}, & \text{if } n \text{ is odd} \\ 2(-1)^{n/2} - (\phi(C_n) + (\phi(C_n))^{-1}), & \text{if } n \text{ is even.} \end{cases}$$

Proof. From Eq. (7), we see that characteristic polynomial of a gain path satisfies $\Psi(\Phi_{P_n}, 0) = 0$ when n is odd and $\Psi(\Phi_{P_n}, 0) = (-1)^{n/2}$ when n is even. Then the result follows from Eq. (8), since $\det(A(\Phi_{C_n})) = (-1)^n\Psi(\Phi_{C_n}, 0)$. $\square$

Corollary 3.7.

$$\det(A(C_n)) = \begin{cases} 2, & \text{if } n \equiv \pm 1 \pmod 4 \\ 0, & \text{if } n \equiv 0 \pmod 4 \\ -4, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Corollary 3.8. *The adjacency matrix of a cycle C_n is singular only when $n \equiv 0 \pmod 4$.*

We denote by $C_n^{(r)}$, a signed cycle with r negative edges on the cycle C_n. In [4], the authors discuss about the eigenvalues of $C_n^{(r)}$ given in the following Lemma. Using that result we have a useful algebraic product of cosines, in view of the above Corollary 3.7, which would otherwise be hard to compute. Also we adopt the notation, $[r]$, for an integer r which means $[r] = 0$, if r is even and $[r] = 1$, if it is odd. In the formula, that follows, for $\det(A(C_n^{(r)}))$, one shall notice the fact that $(-1)^{[r]}$ is actually the product of signs of edges.

Lemma 3.9 [4]. *The eigenvalues $\lambda_j \in \mathbb{R}$ of $C_n^{(r)}$ for $j = 1, 2, \ldots, n$ and $0 \le r \le n$ are given by*

$$\lambda_j = 2\cos\frac{(2j-[r])\pi}{n}$$

Corollary 3.10.

$$\det(A(C_n^{(r)})) = \begin{cases} 2(-1)^{[r]}, & \text{if } n \text{ is odd} \\ 2((-1)^{n/2} - (-1)^{[r]}), & \text{if } n \text{ is even.} \end{cases}$$

Proof. Applying Corollary 3.6 and noting the fact that $C_n^{(r)}$ is balanced when $[r] = 0$, we get the result. □

Corollary 3.11. *For $0 \le r \le n$,*

$$\prod_{j=1}^{n} \cos\frac{(2j-[r])\pi}{n} = \begin{cases} 2^{1-n}(-1)^{[r]}, & \text{if } n \equiv 1 \pmod 2 \\ 2^{1-n}((-1)^{n/2} - (-1)^{[r]}), & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

Proof. Here we note that $\det(A(C_n^{(r)})) = \prod_{j=1}^{n} \lambda_j$ and the proof follows from the previous Lemma 3.9 and the Corollary 3.10. □

4. Further directions

The recurrence relation for the gain graph, when applied to a signed graph Σ, has another implication that when Σ is considered as a gain graph over $GF(3)^{\times}$, the recurrence formula becomes

$$\begin{aligned} \Psi(\Sigma_G, x) = {} & x\Psi(\Sigma_{G-v}, x) - \sum_{w\sim v} \Psi(\Sigma_{G-\{v,w\}}, x) \\ & + \sum_{C\in\mathcal{B}(v)} \Psi(\Sigma_{G-V(C)}, x) - \sum_{C\in\mathcal{B}'(v)} \Psi(\Sigma_{G-V(C)}, x) \end{aligned}$$

because $-2 \equiv 1 \pmod 3$ apart from the relation that is given in the concerned section which is mainly used when the signed graph is considered as a gain graph over $\{1, -1\}$. Thus it opens up many faceted way of looking at one object when we change the underlying field. We hope further that coding theorists will generally be benefitted from our discussion if they examine the gain graphs over the finite fields especially when the group for labeling its edges is $GF(p)^{\times}$ or $GF(p^k)^{\times}$ for some prime p.

Acknowledgements

The authors express their deep sense of gratitude to Professor Thomas Zaslavsky for his valuable suggestions while preparing this paper which helped them to improve its content and style. First author is indebted to the University Grants Commission (UGC) for granting him Teacher Fellowship under UGC's Faculty Development Programme during XI plan.

References

[1] B.D. Acharya, Spectral criterion for cycle balance in networks, J. Graph Theory 4 (1980) 1–11.
[2] D.M. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, third ed., Johann Abrosius Barth Verlag, 1995 (first ed., DeutscherVerlag derWissenschaften, Berlin, 1980; Academic Press, New York, 1980).
[3] K.A. Germina, Shahul Hameed K, On signed paths, signed cycles and their energies, Appl. Math. Sci. 4 (70) (2010) 3455–3466.
[4] K.A. Germina, Shahul Hameed K, Thomas Zaslavsky, On product and line graphs of signed graphs, their eigenvalues and energy, Linear Algebra Appl. 435 (2011) 2432–2450.
[5] M.K. Gill, B.D. Acharya, A recurrence formula for computing the characteristic polynomial of a sigraph, J. Combin. 1 (1980) 68–72.
[6] F. Harary, Graph Theory, Addison Wesley, Reading, MA, 1972.
[7] F. Harary, The determinant of the adjacency matrix of a graph, SIAM Rev. 4 (1982) 202–210.
[8] Joseph J. Rotman, Advanced Modern Algebra, Prentice Hall, 2002.
[9] Richard A. Brualdi, H.J. Ryser, Combinatorial Matrix Theory, Cambridge University Press, 1991.
[10] A.J. Schwenk, Computing the characteristic polynomial of a graph, Graphs Combin., 406 153–172.
[11] T. Zaslavsky, Signed graphs, Discrete Appl. Math. 4 (1982) 47–74, Erratum: Discrete Appl. Math. 5 (1983) 248.
[12] T. Zaslavsky, Matrices in the theory of signed simple graphs, in: B.D. Acharya, G.O.H. Katona, J. Nesetril (Eds.), Advances in Discrete Mathematics and Applications: Mysore, 2008 (Proceedings of the International Conference on Discrete Mathematics, Mysore, India, 2008), pp. 207–229. Ramanujan Mathematical Society Lecture Notes Series, No. 13. Ramanujan Mathematical Society, Mysore, India, 2010.
[13] T. Zaslavsky, Biased graphs. I: Bias, balance, and gains, J. Combin. Theory Ser. B 47 (1989) 32–52.